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Free-surface wave damping due to viscosity and surfactants

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Abstract

The steady wave pattern developed on the free surface in inviscid two-dimensional fluid domain is well known. In this work, we investigate the influence of a thin film of insoluble material on an air–liquid interface on the wave characteristics. We employ a quasi-potential approximation and assume that the main flow is potential, while the effect of viscosity and surfactants enters the problem via Bernoulli equation. Thus, the problem is formulated in terms of a single potential function. The formulation presented in this work can be extended to three-dimensional problems. The wave amplitude consistent with the approximations made, may be moderate. We use our formulation to study a pressure distribution moving with a constant velocity, on a two-dimensional deep water air–liquid interface, on which there exists a monolayer film of surface active agents. The problem is solved numerically by a boundary integral equation method and numerical solutions are presented. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction

An important problem in fluid mechanics is the prediction of the wave pattern generated by an obstacle moving at a constant velocity on a free surface or below it. This problem is usually treated by neglecting viscosity and surface tension. These assumptions are very reasonable in ship hydrodynamics but the effect of surface tension is not negligible for small obstacles such as insects or probes. The presence of ambient surfactants on the ocean free-surface is also known to change the radar backscattering of the Kelvin wave pattern of surface ships and considerably affect their SAR images [1].

The purpose of this paper is to study wave patterns in the presence of surface tension. We take into account the effects of viscosity and surfactants. For simplicity we assume that the flow is two-dimensional. We first derive an appropriate set of boundary conditions on the free surface. These are based on the quasi-potential aprroximation [2,3] to model the weak effect of viscosity and the Boussinesq–Scriven constitutive relationship [4–6] to model the effects of insoluable surfactants with small Gibbs elasticity. The quasi-potential approximation enables us then to assume that the main flow is potential, calculate the free-surface profile and conduct a parameter study regarding the relative importance of the various physical parameters.

These boundary conditions are then used to study the wave pattern generated by a distribution of pressure moving at a constant velocity at the surface of a fluid. The induced wave resistance of pressure patches due to viscosity and surface tension can also be computed by following the analysis in [7]. This can be interpreted as an inverse problem for a two-dimensional obstacle moving at a constant velocity at the surface of a fluid. Here the shape of the obstacle is not known a priori but it is given at the end of the calculations by the shape of the free surface below the distribution of pressure. The problem is reformulated as a system of integro-differential equations. These equations are discretized and the resulting nonlinear algebraic equations are solved by Newton's method. Numerical results are presented for various values of the Reynolds, Boussinesq,

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Marangoni and Weber numbers. In addition, we show that there are multiple solutions for viscous flows with constant surface tension.

The fully non-linear free-surface boundary conditions for the general problem are first derived and discussed in Section 2. The linearized case is further elaborated in Section 3 and some useful expressions for the combined decay rate and the extended dispersion relation are obtained. The linearized solution serves as a basis for the quasi-potential approach presented in Section 4, mainly for estimating the order of magnitude of the various quantities which arise in the full problem. The quasi-potential methodology, for a slightly viscous fluid and small surface elasticity, is finally applied in Section 5 to numerically compute typical free-surface profiles induced by a moving pressure distribution under the combined effect of gravity, viscosity, surface tension and film elasticity.

2. Formulation

2.1. Equations of motion

The Navier–Stokes equations can be written in two equivalent forms:

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = -\frac{\nabla P}{\rho} + \boldsymbol{g} + \nu\nabla^2\boldsymbol{V} \tag{1}$$

or

$$\frac{\partial \boldsymbol{V}}{\partial t} + \nabla\left(\frac{P}{\rho} + gy + \frac{1}{2}V^2\right) = \boldsymbol{V}\times\boldsymbol{\omega} - \nu\nabla\times\boldsymbol{\omega}. \tag{2}$$

In the above, y denotes the vertical distance above an arbitrary horizontal level and $\boldsymbol{V}(x,y,t)$ is the velocity field. The mass density of the liquid is denoted by ρ, g is the gravity acceleration, ν is the kinematic viscosity, $\boldsymbol{\omega}$ is the vorticity and P is the pressure. We express (2) evaluated on the free surface in the component form:

$$\frac{\partial V_s}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial s} + g\sin\theta + V_s\frac{\partial V_s}{\partial s} + V_n\frac{\partial V_n}{\partial n} = -V_n\omega_s \tag{3}$$

and

$$\frac{\partial V_n}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial n} + g\cos\theta + V_s\frac{\partial V_s}{\partial n} + V_n\frac{\partial V_n}{\partial n} - \nu\frac{\partial \omega_s}{\partial s} = \omega_s V_s, \tag{4}$$

where (3) denotes the projection of (2) along the tangent vector $\hat{t}$ to the free surface and (4) represents its projection along the normal vector $\hat{n}$. Here V_s and V_n are the tangential and normal components of the velocity vector on the free surface. In (3) and (4), $\tan\theta$ denotes the free surface slope. The interfacial vorticity vector is given by

$$\boldsymbol{\omega}_s = \left(\frac{\partial V_s}{\partial n} - \frac{\partial V_n}{\partial s} - \kappa V_s\right)\hat{\varphi}, \tag{5}$$

where κ is the free-surface curvature and the unit vector $\hat{\varphi}$ is defined as

$$\hat{t}\times\hat{\varphi} = \hat{n}. \tag{6}$$

Eq. (6) means that the triad $(\hat{t},\hat{\varphi},\hat{n})$ defines an orthogonal, right-handed reference frame. Thus, ω_s in (3) and (4) is identical with the coefficient of $\hat{\varphi}$ in (5). The continuity equation on the free surface is

$$\nabla\cdot\boldsymbol{V} = \frac{\partial V_s}{\partial s} + \frac{\partial V_n}{\partial n} - \kappa V_n = 0. \tag{7}$$

The Bernoulli equation can be written in two equivalent ways:

$$\frac{P}{\rho} + g\eta + \frac{1}{2}\left(V_n^2 + V_s^2\right) = \int_{-\infty}^{\eta}\left((\boldsymbol{\omega}\cdot\hat{\varphi})V_s + \nu\frac{\partial(\boldsymbol{\omega}\cdot\hat{\varphi})}{\partial s} - \frac{\partial V_n}{\partial t}\right)\mathrm{d}n \tag{8}$$

or

$$\frac{P}{\rho} + g\eta + \frac{1}{2}\left(V_n^2 + V_s^2\right) = -\int_{0}^{s}\left(\frac{\partial V_s}{\partial t} + \omega_s V_n - \nu\frac{\partial \omega_s}{\partial n}\right)\mathrm{d}s + \text{const.}, \tag{9}$$

where η denotes the free surface elevation, and s is a distance measured along the free-surface in the streamwise direction.

2.2. Boundary conditions

The full boundary conditions for a Newtonian type interface in the presence of surfactants are given, for example, by Edwards et al. ([4], Chapter 4, or [5], Eqs. 2.5–2.6). The derivation of these boundary conditions is based on the so called Boussinesq–Scriven constitutive relationship (Scriven [6]) which is expressed in terms of two quantities μ^s and κ^s – the interfacial shear and dilatational viscosities, respectively.

The corresponding boundary conditions in the tangential and normal direction to be applied on the free surface, are given respectively by

$$\mu\left(\frac{\partial V_s}{\partial n}+\frac{\partial V_n}{\partial s}+\kappa V_s\right)=\left(\kappa^s+\mu^s\right)\frac{\partial}{\partial s}\left(\frac{\partial V_s}{\partial s}-\kappa V_n\right)+\frac{\partial \sigma}{\partial s}, \tag{10}$$

$$-P+2\mu\frac{\partial V_n}{\partial n}=\kappa\left[\sigma+\left(\kappa^s+\mu^s\right)\left(\frac{\partial V_s}{\partial s}-\kappa V_n\right)\right], \tag{11}$$

where μ is the dynamic viscosity of the bulk and σ is the surface tension coefficient. The quantity in the squared brackets of (11) is an effective surface tension which we denote here by σ_{eff}. The above relations which serve as the basic boundary conditions to be used in the sequel, can also be written as

$$\mu\left(\frac{\partial V_s}{\partial n}+\frac{\partial V_n}{\partial s}+\kappa V_s\right)=\frac{\partial \sigma_{\mathrm{eff}}}{\partial s}, \tag{12}$$

$$-P+2\mu\frac{\partial V_n}{\partial n}=\kappa\sigma_{\mathrm{eff}}. \tag{13}$$

The kinematic condition is next expressed by

$$\dot{\eta}=\frac{V_n}{\cos\theta}, \tag{14}$$

where the dot means partial derivative with respect to the time t. Eq. (14) is a compact form of the kinematic condition, more familiar as

$$\frac{D\eta}{Dt}=\dot{\eta}+(\boldsymbol{V}\cdot\nabla)\eta=0. \tag{15}$$

The vorticity equation at any point in the flow domain is

$$\frac{D\boldsymbol{\omega}}{Dt}=(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{V}-\nu\nabla^2\boldsymbol{\omega}, \tag{16}$$

where the Laplacian operator is defined in terms of the tangential and normal derivatives as

$$\nabla^2=\frac{\partial^2}{\partial s^2}-\kappa(n,s)\frac{\partial}{\partial n}+\frac{\partial^2}{\partial n^2}. \tag{17}$$

Note that the free surface curvature κ depends on the coordinate (n,s). When applied to the free surface, (16) takes the form

$$\frac{\partial \omega_s}{\partial t}+V_s\frac{\partial \omega_s}{\partial s}+V_n\frac{\partial \omega_s}{\partial n}=-\nu\nabla^2\omega_s. \tag{18}$$

The normal and tangential derivatives of ω_s on the free surface are obtained from (5) as

$$\frac{\partial \omega_s}{\partial n}=-\kappa\frac{\partial V_s}{\partial n}-\kappa^2 V_s-\kappa\frac{\partial V_n}{\partial s}-2V_s\frac{\partial \kappa}{\partial n}+\frac{\partial^2 V_s}{\partial n^2}-\frac{\partial^2 V_n}{\partial n\partial s}, \tag{19}$$

$$\frac{\partial \omega_s}{\partial s}=-\kappa\frac{\partial V_s}{\partial s}-V_s\frac{\partial \kappa}{\partial s}+\frac{\partial^2 V_s}{\partial n\partial s}-\frac{\partial^2 V_n}{\partial s^2}. \tag{20}$$

The above three equations will be used later to estimate the surface vorticity ω_s on the free surface.

3. The linear problem

In the previous section the fully non-linear equations and boundary conditions were derived. Here we linearize the problem and obtain order of magnitude estimates of the various flow variables. These estimates will serve as a basis for approximating the non-linear problem.

3.1. The linear equations

By neglecting the convective nonlinear term in (1) we obtain:

$$\frac{\partial \boldsymbol{V}}{\partial t} = -\frac{\nabla P}{\rho} + \boldsymbol{g} + \nu \nabla^2 \boldsymbol{V}. \tag{21}$$

The dynamic boundary conditions (10), (11) reduce respectively to:

$$\mu\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) = \frac{\partial \sigma}{\partial x} + \left(\kappa^s + \mu^s\right)\frac{\partial^2}{\partial x^2} V_x, \tag{22}$$

$$-P - 2\mu \frac{\partial V_x}{\partial x} = \sigma_0 \eta_{xx}, \tag{23}$$

where σ_0 is the equilibrium value of the surface tension. The coordinates (s, n) reduce in the linear case to the ordinary Cartesian coordinates (x, y). The surfactant concentration ρ^s on the free surface must satisfy the diffusion equation

$$\frac{\partial \rho^s}{\partial t} + \nabla_s \cdot \left(\boldsymbol{V} \rho^s\right) = 0, \tag{24}$$

where we have disregarded absorption and flux of material into the insoluble film assumed to exist on the interface.

Let us look for a plane wave solution in the form

$$\rho^s(x, t) = \rho_0^s + R\mathrm{e}^{i(kx - \omega t)}, \tag{25}$$

where k denotes the wavenumber, ω is the wave frequency, ρ_0^s is the equilibrium value of ρ^s assumed constant, and R is a small constant amplitude number. The surface tension is assumed to depend linearly on ρ^s,

$$\sigma = \sigma_0 - \frac{E_0}{\rho_0^s}\left(\rho^s - \rho_0^s\right). \tag{26}$$

The Gibbs elasticity E_0 is defined according to the Langmuir-type model as

$$E_0 = \frac{R_0 T}{M} \rho_\infty^s \left[\frac{\rho_0^s / \rho_\infty^s}{1 - \rho_0^s / \rho_\infty^s}\right], \tag{27}$$

where ρ_∞^s is the surface saturation density, R_0 is the gas constant, T is the absolute temperature and M the molecular density of the film material.

To solve (21), we assume that $\boldsymbol{V}$ has the form (Helmholtz decomposition)

$$\boldsymbol{V} = \nabla \phi + \boldsymbol{v}, \tag{28}$$

where ϕ is the potential and $\boldsymbol{v}$ the rotational part. The potential ϕ satisfies

$$\frac{\partial}{\partial t} \nabla \phi = -\frac{\nabla P}{\rho} + \boldsymbol{g} \tag{29}$$

and consequently the vortical part obeys the equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nu \nabla^2 \boldsymbol{v} \tag{30}$$

together with the incompressibility condition

$$\nabla \cdot \boldsymbol{v} = 0. \tag{31}$$

The analysis appears to be easier when using a stream function ψ satisfying the diffusion equation

$$\frac{\partial \psi}{\partial t} = \nu \nabla^2 \psi. \tag{32}$$

The Cartesian velocity components of $\boldsymbol{V}$ are then expressed in terms of ϕ and ψ as

$$V_x = \phi_x - \psi_y, \tag{33}$$

$$V_y = \phi_y + \psi_x. \tag{34}$$

The linearised kinematic boundary condition and the Bernoulli equation can be written as

$$\frac{\partial \eta}{\partial t} = V_y \tag{35}$$

and

$$\frac{\partial\phi}{\partial t} + g\eta + \frac{P}{\rho} = 0. \tag{36}$$

Finally, we impose a proper decay condition at infinity by demanding that

$$\boldsymbol{V} = 0 \quad \text{at } y \to -\infty. \tag{37}$$

3.2. The dispersion relation

In order to solve the above linear system of equations we assume the following plane wave solutions:

$$\phi(x, y, t) = A\mathrm{e}^{ky+i(kx-\omega t)} + \text{c.c.}, \quad \mathrm{Re}(k) > 0, \tag{38}$$

$$\psi(x, y, t) = B\mathrm{e}^{my+i(kx-\omega t)} + \text{c.c.}, \quad \mathrm{Re}(m) > 0, \tag{39}$$

where A and B are arbitrary complex amplitudes, c.c. denotes complex conjugation and Re denotes the real part. The conditions on the real parts of k, m guarantee decay with depth. The imaginary part of k represents the spatial wave damping.

By substituting (38), (39) into the two dynamic conditions (22), (23) we obtain two coupled equations in A and B:

$$A\left[2\nu k^2 + \frac{(\kappa^s + \mu^s)}{\rho}k^3 + \frac{ik^3E_0}{(\rho\omega)}\right] + B\left[i\nu\left(m^2 + k^2\right) + im\frac{(\kappa^s + \mu^s)}{\rho}k^2 - \frac{mE_0k^2}{\rho\omega}\right] = 0, \tag{40}$$

and

$$A\left[\omega - \frac{gk}{\omega} - \frac{\sigma_0 k^3}{\rho\omega} + 2i\nu k^2\right] - B\left[\frac{igk}{\omega} + 2\nu km + i\frac{\sigma_0 k^3}{\rho\omega}\right] = 0, \tag{41}$$

where m is determined from (32) as

$$m^2 = k^2 - \frac{i\omega}{\nu}. \tag{42}$$

The constant R in (25) is found by substituting (25) into (24),

$$R = \frac{i(k^2A + ikmB)\rho_0^s}{\omega}. \tag{43}$$

The dispersion relation is found next by requiring that the determinant of the coefficients A, B in (40), (41) must vanish. Denoting for short

$$u \equiv \frac{\omega}{\Omega} \tag{44}$$

with

$$\Omega^2 = gk + \sigma_0 k^3, \tag{45}$$

we obtain from (40), (41):

$$A\left[2\delta u + \frac{k^3}{\rho\Omega}\left(\left(\kappa^s + \mu^s\right)u + \frac{iE_0}{\Omega}\right)\right] + B\left[u^2 + 2i\delta u + \frac{im}{k}\cdot\frac{k^3}{\rho\Omega}\left(\left(\kappa^s + \mu^s\right)u + \frac{iE_0}{\Omega}\right)\right] = 0 \tag{46}$$

and

$$A\left[-1 + 2i\delta u + u^2\right] - B\left[i + 2\sqrt{-i\delta}\,u^{3/2}\right] = 0. \tag{47}$$

The quantity δ in the above is defined by

$$\delta = \frac{\nu k^2}{\Omega} \tag{48}$$

and is a measure of the boundary layer thickness. As a check, we note that when surfactants are not present one obtains the classical result [8]

$$u = 1 - 2i\delta. \tag{49}$$

To solve for u, it is assumed that the surfactant introduce only a small change in (49), i.e.

$$u = 1 - 2i\delta + \gamma, \tag{50}$$

where γ is a complex number with small modulus

$$|\gamma| \ll 1. \tag{51}$$

The smallness of $|\delta|$ and $|\gamma|$ enables us to neglect the square terms in expanding the determinant of (46), (47). We then find

$$\gamma = \frac{-\frac{i}{2}\theta_1}{1 + \frac{\sqrt{2}}{2}(1+i)\frac{\theta_1}{\sqrt{\delta}}}, \tag{52}$$

where θ_1 is defined as

$$\theta_1 = \frac{k^3}{\rho\Omega}\left[\frac{iE_0}{\Omega} + \kappa^s + \mu^s\right], \tag{53}$$

which together with (44), (48) and (50) determine the dispersion relation.

3.3. *The decay rate*

We wish next to explore the conditions under which (51) is satisfied. For this purpose, let us first define a complex variable z as

$$z = \frac{\theta_1}{\sqrt{\delta}}. \tag{54}$$

Then, $|\gamma|$ expressed as a function of z, can be written as

$$|\gamma| = \frac{1}{2}\left|\sqrt{\delta}\right|\frac{|z|}{|1 + \mathrm{e}^{i\pi/4}z|} \tag{55}$$

or

$$|\gamma| = \frac{|\sqrt{\delta}|}{2}\frac{1}{|\frac{1}{z} + \mathrm{e}^{i\pi/4}|}. \tag{56}$$

Define

$$g(z) = \left|\frac{1}{z} + \mathrm{e}^{i\pi/4}\right|^2. \tag{57}$$

Then the condition (51) reads

$$|\gamma| = \frac{|\sqrt{\delta}|}{2}\frac{1}{\sqrt{g(z)}} \ll 1 \tag{58}$$

or

$$g(z) \gg \frac{|\delta|}{4}. \tag{59}$$

Expressing z in polar coordinates $z = R\mathrm{e}^{i\theta}$; then $g(z)$ can be written as

$$g(z) = 1 + \frac{1}{R^2} + \frac{\sqrt{2}}{R}(\cos\theta - \sin\theta). \tag{60}$$

The function $g(z)$ attains a minimum at the point

$$z_0 = \mathrm{e}^{i3\pi/4} \tag{61}$$

with

$$g_{\min} \equiv g(z_0) = 0. \tag{62}$$

At the point z_0, $|\gamma|$ becomes singular and obviously (51) is not satisfied there. The function $g(z)$ has the value $|\delta|/4$ on the boundary of a region around the point z_0 whose dimension scales as $|\sqrt{\delta}|$. On the boundary $|\gamma| = 1$, and outside this region $g(z)$ becomes gradually O(1) and $|\gamma|$ becomes O($|\sqrt{\delta}|$). Thus, the condition (51) is satisfied whenever the point z under consideration resides far from the points of the above region. A rough criteria for this is

$$|z - z_0| \gg |\sqrt{\delta}|. \tag{63}$$

We shall prove below that (63) is satisfied provided $|\delta|$ is small enough. In the application of Section 5, we assume that the wave under consideration is steady in a frame of reference moving with the pressure disturbance. Therefore we may write

$$\omega = kc. \tag{64}$$

Using the dispersion relation, which is valid for small $|\delta|$ and small $|\gamma|$, we obtain the following equation for the complex wave number k:

$$kc = \Omega(k)\big[1 - 2i\delta(k) + \gamma(k)\big]. \tag{65}$$

By decomposing γ and δ into their real and imaginary components, i.e.

$$\gamma = \gamma_0 + i\gamma_1, \tag{66}$$

$$\delta = \delta_0 + i\delta_1, \tag{67}$$

we obtain from (65) two real equations for k_0 and k_1 ($k = k_0 + ik_1$):

$$ck_0 = \Omega_0(1 + \gamma_0 + 2\delta_1), \tag{68}$$

$$ck_1 = \Omega_0(\gamma_1 - 2\delta_0) + k_1 c_{g0}(1 + \gamma_0 + 2\delta_1), \tag{69}$$

where c_{g0} is the so-called 'inviscid' group velocity defined as

$$c_{g0} = \frac{\partial}{\partial k_0}\Omega(k_0) = \frac{g + 3\sigma_0 k_0^2}{2\Omega_0} \tag{70}$$

and

$$\Omega_0 = \big(gk_0 + \sigma_0 k_0^3\big)^{1/2}. \tag{71}$$

It should be noted that for a viscous fluid the group velocity is not equal to the energy velocity but remains perpendicular to the wave vector in a similar manner to the corresponding conservative case. The difference between these two velocities is a result of viscous dispersion [9]. To lowest order we obtain from (68) and (69) that

$$ck_0 = \Omega_0 \tag{72}$$

and

$$\frac{k_1}{k_0} = \frac{c}{c - c_{g0}}(\gamma_1 - 2\delta_0). \tag{73}$$

Substituting Ω_0 from (71) in (72), we obtain an explicit expression for k_0 in terms of the known quantities:

$$k_0 = \frac{c^2 \pm \sqrt{c^4 - 4\sigma_0 g}}{2\sigma_0}. \tag{74}$$

The value $(4\sigma_0 g)^{1/4}$ is the minimum value of the phase speed of a capillary gravity linear wave. It should be noted that the determination of a numerical value of c and the requirement of steady state are not independent. According to (74), a steady wave of constant amplitude cannot be realized for $c^4 < 4\sigma_0 g$ and for fixed k_0. Alternatively for fixed (c, σ_0, g), Eq. (72) does not admit any real solution for k_0.

Eq. (73) shows that the decay constant k_1, which determines the wave decay, is small if $|\delta|$ is small. The smallness of $|\delta|$ is assured once (63) holds. Then, we can write θ_1 from (53) as

$$\theta_1 = \frac{k_0^2}{\rho c}\left[\frac{iE_0}{k_0 c} + \kappa^s + \mu^s\right] + \mathrm{O}\left(\frac{k_1}{k_0}\right) \tag{75}$$

and the complex number δ can be approximated by its real part

$$\delta_0 = \frac{\nu k_0^2}{\Omega_0} + \mathrm{O}\left(\frac{k_1}{k_0}\right). \tag{76}$$

From (75) it is clear that θ_1 is a complex number in the first quadrant and since δ according to (76) is real, we deduce from the definition of z (Eq. (54)) that z resides in the first quadrant as well. From the definition of z_0 (Eq. (61)) it is obvious that the condition (63) is also satisfied. We have thus demonstrated that $|\delta| \ll 1$, automatically guarantees that $|\gamma| \ll 1$ and $|k_1/k_0| \ll 1$ as well. The above conjectures are independent of the surface concentration of the surfactant ρ_0^s, i.e. it is true for any value of E_0 and of $(\kappa^s + \mu^s)$.

The expression for γ in light of the above discussion can be thus written as

$$\gamma = \frac{-i}{2}\sqrt{\delta_0}\cdot\frac{z}{1+e^{i\pi/4}z}, \tag{77}$$

where z is approximated by

$$z = \frac{\theta_1}{\sqrt{\delta_0}} \tag{78}$$

and θ_1 is given by (75). The maximum value of $|\gamma|$ can then be easily found.

From the theory of complex variables it is known that $|\gamma(z)|$ attains extreme values on the boundary of the first quadrant. Elementary calculations reveal that

$$|\gamma|_{\max} = \frac{\sqrt{\delta_0}}{2} \tag{79}$$

which happens when $|z| \to \infty$. Using the definition of z (Eq. (78)), and the fact that both E_0 and $(\kappa^s + \mu^s)$ can be considered as increasing monotonic functions of ρ_0^s, it is realized that the maximum value of $|\gamma|$ as given by (79), occurs when $\rho_0^s \to \infty$. Thus,

$$\gamma\left(\rho_0^s \to \infty\right) = \frac{-\sqrt{2}}{4}(1+i)\sqrt{\delta_0}, \tag{80}$$

where (77) has been used to derive (80). The above result is in agreement with Levich ([8], Eq. 121.32). The real and imaginary parts of γ (Eq. (77)) are found as

$$\mathrm{Re}(\gamma) = \gamma_0 = \frac{\frac{\sqrt{\delta_0}}{2}\left[y_1 - \frac{\sqrt{2}}{2}\left(x_1^2 + y_1^2\right)\right]}{1 + x_1^2 + y_1^2 + \sqrt{2}(x_1 - y_1)} \tag{81}$$

and

$$\mathrm{Im}(\gamma) = \gamma_1 = \frac{\frac{-\sqrt{\delta_0}}{2}\left[x_1 + \frac{\sqrt{2}}{2}\left(x_1^2 + y_1^2\right)\right]}{1 + x_1^2 + y_1^2 + \sqrt{2}(x_1 - y_1)}, \tag{82}$$

where x_1, y_1 represent the real and imaginary parts of θ_1 (Eq. (75)), i.e.

$$x_1 = \frac{k_0^2}{\rho c}\left(\kappa^s + \mu^s\right) \tag{83}$$

and

$$y_1 = \frac{k_0}{\rho c^2}E_0. \tag{84}$$

From Eq. (82) it is seen that $\mathrm{Im}(\gamma) < 0$ from which we conclude next (see Eq. (50)) that the presence of surfactants enhance the wave decay.

The dispersion relation (50) can be finally written as

$$\frac{\omega}{\Omega} = (1+\gamma_0) - 2i\left(\delta_0 - \frac{\gamma_1}{2}\right). \tag{85}$$

From the above equation it may be seen that δ_0 and $-\gamma_1/2$ (which is positive) play the same role and thus an effective viscosity can be defined as

$$\nu_{\mathrm{eff}} = \nu + \frac{|\gamma_1|}{2}\frac{c}{k_0}. \tag{86}$$

The dispersion relation (85) is illustrated graphically in Figs. 1 and 2, where we present values of the real and imaginary parts of ω/Ω versus x_1 for $y_1 = 0.5$ and various values of δ. An expression for k_1 (correct to the first order) can be written from (73) as

$$\frac{k_1}{k_0} = \frac{-2c^2[|\gamma_1| + 2\delta_0]}{gk_0 - \sigma_0 k_0^3}, \tag{87}$$

which is always negative for $\sigma_0 k_0^2/g < 1$. When c approaches c_g, non-linear effects must clearly be taken into account.

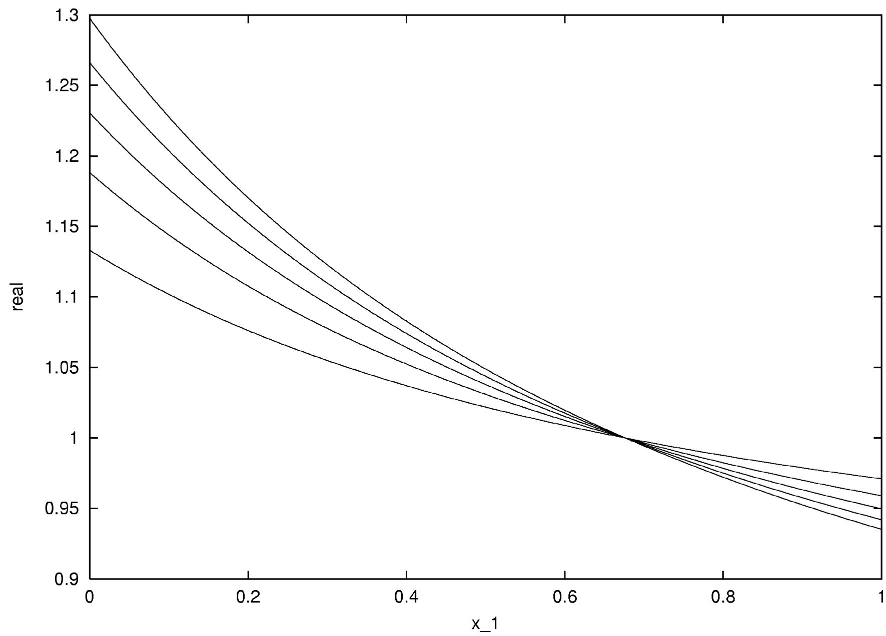

Fig. 1. Real part of ω/Ω versus x_1 for $y_1 = 0.5$ and various values of δ. The curves in order of increasing vertical intercepts correspond to $\delta = 0.2$, $\delta = 0.4$, $\delta = 0.6$, $\delta = 0.8$ and $\delta = 1$.

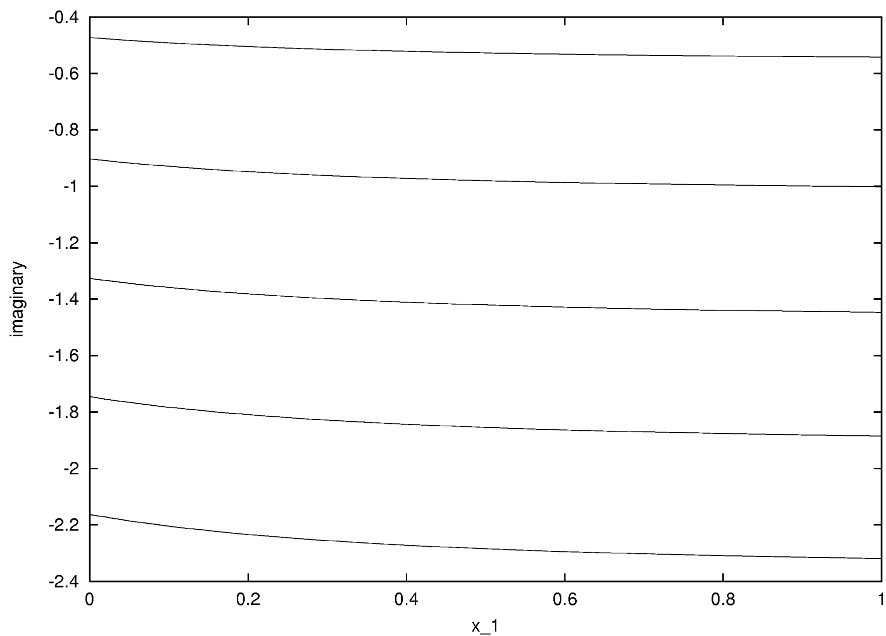

Fig. 2. Imaginary part of ω/Ω versus x_1 for $y_1 = 0.5$ and various values of δ. The curves from top to bottom correspond to $\delta = 0.2$, $\delta = 0.4$, $\delta = 0.6$, $\delta = 0.8$ and $\delta = 1$.

3.4. The surface vorticity

Next, we wish to derive explicit expressions for the various flow quantities such as the velocities and the surface vorticity. From (46), (47) one gets

$$\frac{B}{A} = -2(\delta_0 + i\gamma). \tag{88}$$

Using (33), (34), (38), (39), (42) and (88) we find that

$$v_x = \frac{2\varepsilon c}{\sqrt{\delta_0}}|\delta_0 + i\gamma|e^{\beta y}\cos\left(k_0 x - \Omega_0 t - \beta y + \alpha - \frac{3}{4}\pi\right), \tag{89}$$

$$v_y = -2\varepsilon c|\delta_0 + i\gamma|e^{\beta y}\cos(k_0 x - \Omega_0 t - \beta y + \alpha), \tag{90}$$

where ε is the wave steepness, and β is a parameter defined as

$$\beta = \frac{\sqrt{2}}{2}\frac{k_0}{\sqrt{\delta_0}}. \tag{91}$$

The angle α is the argument of $(\delta_0 + i\gamma)$, i.e.

$$\alpha = \arg(\delta_0 + i\gamma). \tag{92}$$

The potential velocity components are

$$\phi_x = \varepsilon c e^{k_0 y}\cos(k_0 x - \Omega_0 t), \qquad \phi_y = \varepsilon c e^{k_0 y}\sin(k_0 x - \Omega_0 t). \tag{93}$$

The surface tension is given by

$$\sigma(x) = \sigma_0 - \varepsilon E_0 \mathrm{Re}\left[\left(1 - \frac{e^{i\pi/4}(\delta_0 + i\gamma)}{\sqrt{\delta_0}}\right)e^{i(k_0 x - \Omega_0 t)}\right] \tag{94}$$

and the linearized surface vorticity is

$$\omega_s = \frac{\partial v_x}{\partial y} = \frac{-2\varepsilon\Omega_0|\delta_0 + i\gamma|}{\delta_0}\cos(k_0 x - \Omega_0 t + \alpha). \tag{95}$$

The relative magnitude of the vortical velocity with respect to the potential one is

$$\left|\frac{\boldsymbol{v}}{\nabla\phi}\right| \approx \frac{2|\delta_0 + i\gamma|}{\sqrt{\delta_0}} \tag{96}$$

and by combining (96) and (79) we arrive at the result

$$\left|\frac{\boldsymbol{v}}{\nabla\phi}\right| \leqslant 1. \tag{97}$$

Thus, the vortical velocity vector can reach the same magnitude as the potential velocity vector. This happens when $\rho_0^s \to \infty$. If, on the other hand, $\rho_0^s \to 0$, we obtain from (96):

$$\left|\frac{\boldsymbol{v}}{\nabla\phi}\right| \approx 2\sqrt{\delta_0}. \tag{98}$$

Finally, the free surface elevation is given by

$$\eta(x,t) = \frac{\varepsilon}{k_0}\cos(k_0 x - \Omega_0 t) + O\big(\varepsilon|\delta_0 + i\gamma|\big). \tag{99}$$

As a check on the above results, we note that when $\rho_0^s \to \infty$ then $\alpha \to -\pi/4$ and the total horizontal velocity $(\phi_x + v_x)$ vanishes, in agreement with Levich ([8], Eq. 121.34).

4. The quasi-potential approximation

In several papers Ruvinsky and Friedman (see [3] and the references cited there) have formulated a theory for analyzing slightly viscous capillary-gravity ripples riding on the front of a longer gravity wave. These authors, according to the well known Helmholtz decomposition, separated the velocity field into a potential part and an rotational one. The resulting coupled equations enable one to solve the potential part and deduce from it the (assumed small) irrotational one. Longuet-Higgins [2] gave a simpler form of the equations by applying the boundary conditions on a slightly displaced interface. In this section we apply the Helmholtz decomposition to the full equations and analyze the various results. The ordering of the various quantities such as the velocity components, will rely on the results from our previous analysis of the linear theory. One can easily verify that the quasi-potential approximation is valid for a slightly viscous fluid with weak film elasticity that is where the two parameters defined by (48) and (52) are small compared to unity.

We begin by considering Eqs. (8) and (28). Thus we write (9) as

$$\frac{P}{\rho}+\frac{\partial\phi}{\partial t}+g\eta+\frac{1}{2}\left(V_n^2+V_s^2\right)=\int_{-\infty}^{\eta}\left((\boldsymbol{\omega}\cdot\hat{\varphi})V_s+\nu\frac{\partial(\boldsymbol{\omega}\cdot\hat{\varphi})}{\partial s}-\frac{\partial V_n}{\partial t}\right)\mathrm{d}n. \tag{100}$$

The integral on the right-hand side of (100) is of the order $\varepsilon\delta_0^{3/2}$, thus the right-hand side of (100) can be neglected whenever

$$\left|\frac{\delta_T}{\sqrt{\delta_0}}\right|\ll 1, \tag{101}$$

where δ_T is defined as

$$\delta_T=\delta_0+i\gamma. \tag{102}$$

The dynamic pressure can then be written as

$$P=\kappa\sigma+\left[2\mu+\kappa\left(\kappa^s+\mu^s\right)\right]\left(\phi_{nn}+\frac{\partial v_n}{\partial n}\right), \tag{103}$$

where Eqs. (10), (11) and (7) have been used. According to the previous estimates,

$$\frac{\partial v_n}{\partial n}\sim\varepsilon\frac{|\delta_T|}{\sqrt{\delta_0}} \tag{104}$$

and

$$\phi_{nn}\sim\varepsilon. \tag{105}$$

Thus (103) can be rewritten as

$$P=\kappa\sigma+\left[2\mu+\kappa\left(\kappa^s+\mu^s\right)\right]\phi_{nn}. \tag{106}$$

Now, since

$$V_n=\phi_n+\mathrm{O}\left(\varepsilon|\delta_T|\right) \tag{107}$$

and

$$V_s=\phi_s+\mathrm{O}\left(\varepsilon\frac{\delta_T}{\sqrt{\delta_0}}\right), \tag{108}$$

we may write (100) as an equation containing potential quantities only:

$$\frac{\partial\phi}{\partial t}+g\eta+\frac{1}{2}(\nabla\phi)^2+\frac{\kappa\sigma}{\rho}+\left[2\nu+\kappa\frac{(\kappa^s+\mu^s)}{\rho}\right]\phi_{nn}=\text{const.} \tag{109}$$

The surface tension in (109) still unknown, shall be determined as a solution of an equation derivable from the continuity equation (24). In the remaining part of this section, we assume that the flow is steady in a frame of reference moving to the left with constant horizontal velocity c. Therefore the variables x and t appear in the combination $x-ct$. By substituting (26) in (24) we obtain:

$$\frac{\partial\sigma}{\partial t}+V_s\frac{\partial\sigma}{\partial s}+(\sigma-\sigma_c)\frac{\partial V_s}{\partial s}=0, \tag{110}$$

where σ_c is the clean interface value of σ given by

$$\sigma_c=\sigma_0+E_0, \tag{111}$$

where, as is recalled, σ_0 is the equilibrium value of σ and E_0 is the Gibbs elasticity, both depend on ρ_0^s. Enforcing the boundary condition

$$\sigma(s=-\infty)=\sigma_0, \tag{112}$$

the exact solution of (110) is derived:

$$\sigma(s)=\sigma_c-E_0\exp\left[\int_{-\infty}^{s}\frac{(\partial V_s/\partial s)\cos\theta}{c-V_s\cos\theta}\,\mathrm{d}s\right], \tag{113}$$

where $\tan\theta$ is the wave slope. Since $\theta \sim \varepsilon$, this equation can be further simplified as

$$\sigma(s) = \sigma_c - E_0 c\left[\frac{1}{c-\phi_s} + \mathrm{O}\left(\frac{\varepsilon|\delta_T|}{\sqrt{\delta_0}}\right)\right] \tag{114}$$

and (109) yields

$$\frac{\partial\phi}{\partial t} + g\eta + \frac{1}{2}(\nabla\phi)^2 + \frac{\kappa}{\rho}\left[\sigma_c - \frac{E_0 c}{c-\phi_s}\right] - \left[2\nu + \kappa\frac{(\kappa^s+\mu^s)}{\rho}\right]\phi_{ss} = \text{const.} \tag{115}$$

The surface vorticity can be found from (5) and (10) as

$$\omega_s = \frac{1}{\mu}\left[\frac{-E_0 c}{(c-\phi_s)^2}\phi_{ss} + \left(\kappa^s+\mu^s\right)\phi_{sss}\right] + 2c\kappa + \mathrm{O}\left(\frac{\varepsilon|\delta_T|}{\delta_0^{3/2}}\right). \tag{116}$$

Note that, the term $2c\kappa$ in (116) is independent of surfactant concentration and depends on the wave geometry only. Thus, it means that vorticity always exists on the wave surface (see Longuet-Higgins [2]).

5. Waves generated by a moving pressure distribution

5.1. Formulation

In this section, we use the equations derived in Section 4 for analyzing the free surface flow generated by a moving distribution of pressure and by accounting for the presence of surfactants and viscosity.

For the purpose of illustration we consider a two-dimensional distribution of pressure $\widetilde{P}$ moving to the left at a constant velocity c at the surface of a fluid of infinite depth and constant density ρ. We seek steady solutions in a frame of reference moving with the pressure distribution. We choose Cartesian coordinates with the y-axis directed vertically upwards and we introduce dimensionless variables by choosing c as the unit velocity and c^2/g as the unit length, where g is the acceleration of gravity. In this section, we denote by Φ the dimensionless velocity potential measured in the steadily moving frame of reference. The transformation between the laboratory and moving coordinate systems imply that

$$\frac{\partial\Phi}{\partial S} = \frac{1}{c}\frac{\partial\phi}{\partial s} - 1,$$

where S is the dimensionless streamwise coordinate.

We use (115) to write the free surface condition, in dimensionless variables, as

$$\frac{1}{2}\left(\frac{\partial\Phi}{\partial S}\right)^2 + y + \widetilde{P} - \frac{2}{Re}\frac{\partial^2\Phi}{\partial S^2}(1+KBo) + K\left(We + \frac{Mr}{\partial\Phi/\partial S}\right) = \frac{1}{2}, \tag{117}$$

where

$$Re = \frac{c^3}{g\nu} \tag{118}$$

is the Reynolds number,

$$Bo = g\frac{\kappa_s+\mu_s}{2\rho\nu c^2} \tag{119}$$

the Boussinesq number,

$$We = g\frac{\sigma_0}{\rho c^4} \tag{120}$$

the Weber number and

$$Mr = gE_0/\rho c^4 \tag{121}$$

the Marangoni number. Here K is the curvature of the free surface and

$$We + \frac{Mr}{\partial\Phi/\partial S} \tag{122}$$

is the dimensionless variable surface tension.

Next we introduce, in addition to the potential function Φ, the streanfunction Ψ. Without loss of generality, we chose $\Psi = 0$ on the free surface. Following Asavanant and Vanden-Broeck [10], it can be shown that

$$x'(\Phi) - 1 = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y'(\varphi)}{\varphi - \Phi} \, \mathrm{d}\varphi. \tag{123}$$

Here we use Φ and Ψ as independent variables and denote by $x(\Phi)$ and $y(\Phi)$ the values of x and y on the free surface $\Psi = 0$. The prime denotes derivative with respect to Φ and the integral in (123) is a Cauchy principal value.

We then express all the derivatives in (117) in terms of $x(\Phi)$ and $y(\Phi)$. This can easily be done by noting that

$$\frac{\partial \Phi}{\partial S} = \left(x'^2 + y'^2\right)^{-1/2}, \tag{124}$$

$$\frac{\partial^2 \Phi}{\partial S^2} = -\frac{x'x'' + y'y''}{(x'^2 + y'^2)^2}, \tag{125}$$

$$K = -\frac{x'y'' - y'x''}{x'^2 + y'^2}. \tag{126}$$

For given values of *Re*, *Bo*, *We* and *Mr*, relations (117), (123)–(126) define a system of integro-differential equations for the two unknown functions $x(\Phi)$ and $y(\Phi)$. This system is discretized and the resulting algebraic equations are solved by Newton's iterations. The discretization is similar to the one used in [10] and the details will not be repeated here. The results presented here are qualitatively independent of the particular distribution of pressure. In our calculation we chose

$$\widetilde{P} = a \exp\left(-b\Phi^2\right). \tag{127}$$

Here a and b are prescribed constants.

Once $x(\Phi)$ and $y(\Phi)$ are known, the non-dimensional free-surface vorticity ω_s is determined from (116):

$$\omega_s = 2K + 2Bo \frac{\partial^3 \Phi}{\partial S^3} - Re\, Mr \frac{\partial^2 \Phi / \partial S^2}{(\partial \Phi / \partial S)^2}. \tag{128}$$

In (128), K and $\partial^2 \Phi / \partial S^2$ are given by (126) and (125) and $\partial^3 \Phi / \partial S^3$ is calculated by

$$\frac{\partial^3 \Phi}{\partial S^3} = -\left[\frac{x'x'' + y'y''}{(x'^2 + y'^2)^2}\right]' \left(x'^2 + y'^2\right)^{-1/2}. \tag{129}$$

Combining $x(\Phi)$ and (127), we also obtain the pressure P as a function of x.

5.2. Numerical results

We used the numerical scheme described in the previous section to compute solutions for various values of *Re*, *Bo*, *We*, Mr, a and b. The results were found to be qualitatively independent of a and b when the parameter a is sufficiently small. Therefore all the results presented here are restricted to the values $a = 0.01$, $a = 0.02$ and $b = 0.3$.

We first examine solutions in the absence of surfactants (i.e. for $Bo = 0$ and $Mr = 0$). Typical free surface profiles are shown in Figs. 3 and 4. The free surface profiles are qualitatively similar, in the sense that they are all characterized by decaying oscillations in the far field. However the decay is only due to viscosity in Fig. 3, whereas it is independent of viscosity for the solid curve of Fig. 4 which corresponds to an inviscid solution.

To explain this point and to interpret the results further, we first recall the linear dispersion relation

$$C^2 = \frac{g\lambda}{2\pi} + \frac{\sigma}{\rho} \frac{2\pi}{\lambda} \tag{130}$$

of gravity capillary waves in the abscence of viscosity and surfactants (see (71)). Here λ is the wavelength and C the phase velocity. Since a is small and the flow is steady, we can expect the waves in the far field (if they are present) to satisfy (130) with $C = c$.

Using our dimensionless variables, we rewrite (130) as

$$1 = \frac{\lambda}{2\pi} + \alpha \frac{2\pi}{\lambda}, \tag{131}$$

where

$$\alpha = \frac{\sigma g}{\rho c^4}.$$

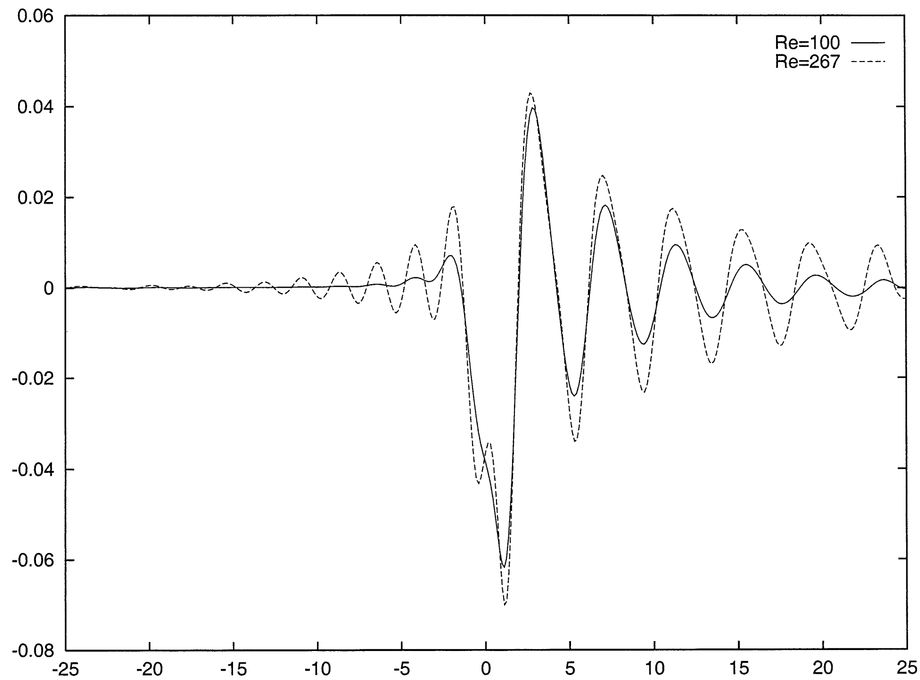

Fig. 3. Free surface profiles with $Re = 100$ and $Re = 267$ for $\alpha = 0.23$, $a = 0.02$, $b = 0.3$, $Bo = 0$ and $Mr = 0$.

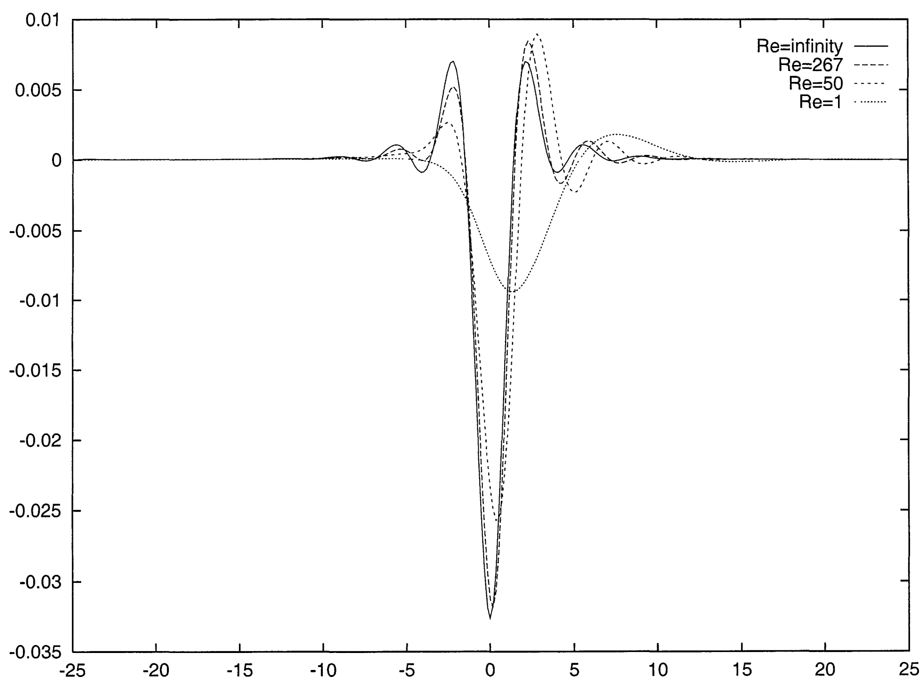

Fig. 4. Free surface profiles with $Re = \infty$, $Re = 267$, $Re = 50$ and $Re = 1$ for $\alpha = 0.23$, $a = 0.01$, $b = 0.3$, $Bo = 0$ and $Mr = 0$.

Solving (131) for λ gives

$$\lambda = \pi \pm \pi(1 - 4\alpha)^{1/2}. \tag{132}$$

For $\alpha < 1/4$, (132) shows that there two real values of λ and the flow in the far field is then a linear combination of

$$\exp\left[i\frac{2}{1 \pm (1-4\alpha)^{1/2}}x\right]. \tag{133}$$

Using the radiation condition we require the solution with the $+$ sign to occur as $x \to \infty$ and the one with the $-$ sign to occur as $x \to -\infty$.

For $\alpha > 1/4$, (132) shows that the two solutions for λ are complex conjugates and the flow in the far field is then approximated by a linear combination of

$$\mathrm{e}^{\frac{ix}{2\alpha}}\mathrm{e}^{[\pm(4\alpha-1)^{1/2}\frac{x}{2\alpha}]}. \tag{134}$$

Requiring the elevation of the free surface to be bounded in the far field, we see that the solution with the $+$ sign occurs as $x \to -\infty$ and the one with the $-$ sign as $x \to \infty$.

The solution (132) corresponds to sine waves of constant amplitude, whereas (133) describe decaying oscillatory tails. This is illustrated in Fig. 3 where the rate of decay is shown to decrease as Re increases (for $Re = \infty$, the waves are of constant amplitude in the far field). On the other hand, the decay survives for $Re = \infty$ when $\alpha > 1/4$ (see Fig. 4). We note that the numerical solutions presented here are nonlinear. However they are close to linear solutions because we chose small values of a.

The linear problem of gravity capillary waves generated by a moving distribution of pressure in the abscence of viscosity was first considered by Rayleigh [11]. His solution behaves like (133) for $\alpha < 1/4$ and like (134) for $\alpha > 1/4$ in accordance with the previous discussion. However Rayleigh's linear solution predicts unbounded displacements of the free surface as $\alpha \to 1/4$. Therefore a nonlinear theory must be used to describe the solutions for α close to $1/4$. This was done numerically in [12] and [13] where it is shown that, as one follows the branch of solutions which behaves like (134) in the far field, the values of α first decreases then reaches a minimum value (greater than $1/4$) and then increases again. Therefore for some values of α, there are two solutions. One is a perturbation of a uniform stream whereas the other is a perturbation of a solitary wave with decaying oscillatory tail. The existence of a minimum value of α on the branch of solutions implies that the solution which behaves like (134) in the far field are not connected to the solutions which behave like (133) in the far field. These two types of solutions form separate branches of solutions and there are no continous transition from one to the other as α is decreased from values greater than $1/4$ to values smaller than $1/4$.

Interesting phenomena near $\alpha = 1/4$, can be expected when the effect of viscosity is introduced. This is suggested by the fact that there are no solitary waves with decaying oscillatory tails in the presence of viscosity (waves in the presence of viscosity cannot travel at a constant velocity without decaying). Therefore we can expect the point at which α reaches a minimum along the inviscid branch of solutions to disappear when viscosity is included and a continuous transition between solutions with $\alpha > 1/4$ and $\alpha < 1/4$ might be possible. This is confirmed by our numerical results. In Fig. 5, we present numerical values of

$$A = -y(0)$$

versus α for $Re = \infty$, $Re = 700$ and $Re = 200$. The parameter A is a measure of the magnitude of the free surface displacement. The curve for $Re = \infty$ agrees with the inviscid nonlinear results of Vanden-Broeck and Dias [12]. As shown by Vanden-Broeck and Dias [12], this curve corrects Rayleigh's linear theory which predicts a vertical asymptote at $\alpha = 1/4$. The numerical results in [12] show that the upper part of the curve for $Re = \infty$ extends to higher values of α and A up to a limiting configuration where the free surface profile touches itself and a trapped bubble appears at the trough. This illustrates the fact that, for $Re = \infty$, the solutions for $\alpha > 1/4$ are not connected to the solutions with $\alpha < 1/4$ (the branch of solutions for $Re = \infty$ shown in Fig. 5 exist only for $\alpha > 0.259$). On the other hand, the corresponding branches for $Re = 200$ and $Re = 700$ extend smoothly from $\alpha > 1/4$ to $\alpha < 1/4$. The value $\alpha = 1/4$ is no longer a critical value and the solutions are qualitatively similar for all values of α: the free surface profiles are characterised by decaying oscillatory tails in the far field. This is to be contrasted to the case $Re = \infty$ where the oscillations in the far field have constant amplitude for $\alpha < 1/4$ and decaying amplitude for $\alpha > 1/4$. We note that the curve for $Re = 700$ in Fig. 5 is closer to the curve for $Re = \infty$ than the curve for $Re = 200$. Therefore the results for $Re \neq \infty$, converge to those for $Re = \infty$ as Re increases.

We note that as Re increases the curves in Fig. 5 become more complicated and several solutions are possible for some values of α (for example for for $Re = 700$ and α close to 0.28).

Next we examine the effects of surfactants. The results for $Mr \neq 0$ and $Bo \neq 0$ are qualitatively similar to those for $Mr = 0$ and $Bo = 0$. However the rates of damping of the waves depend on the values of Mr and Bo. We first illustrate the effect of the Marangoni number by showing in Fig. 6, solutions for $Mr = 0$ and $Mr = 0.05$. As the Marangoni number increases, the effective surface tension decreases and the ripples on the front part of the pressure distribution are less pronounced. The effects of the Boussinesq number are shown in Fig. 7. Here we present results for $Bo = 20$ and $Re = 267$. As the Boussinesq number increases, the damping of the waves increases [14]. This implies that the amplitude of the waves is smaller at a given value of x as Bo increases. Similar results were obtained for other values of Bo and Mr.

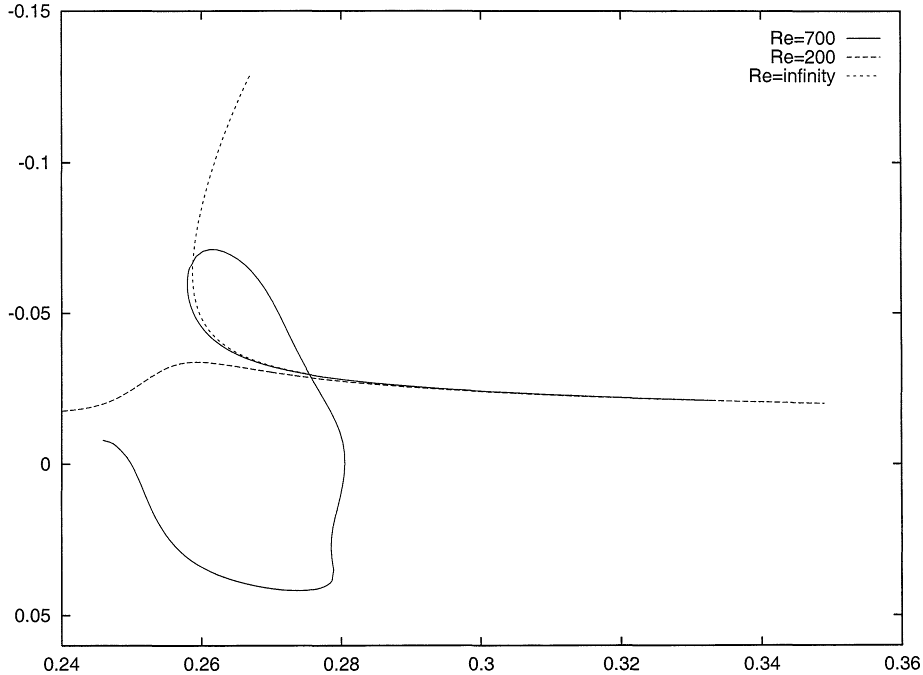

Fig. 5. Values of the amplitude parameter A versus α for $a = 0.01$, $b = 0.3$, $Bo = 0$ and $Mr = 0$.

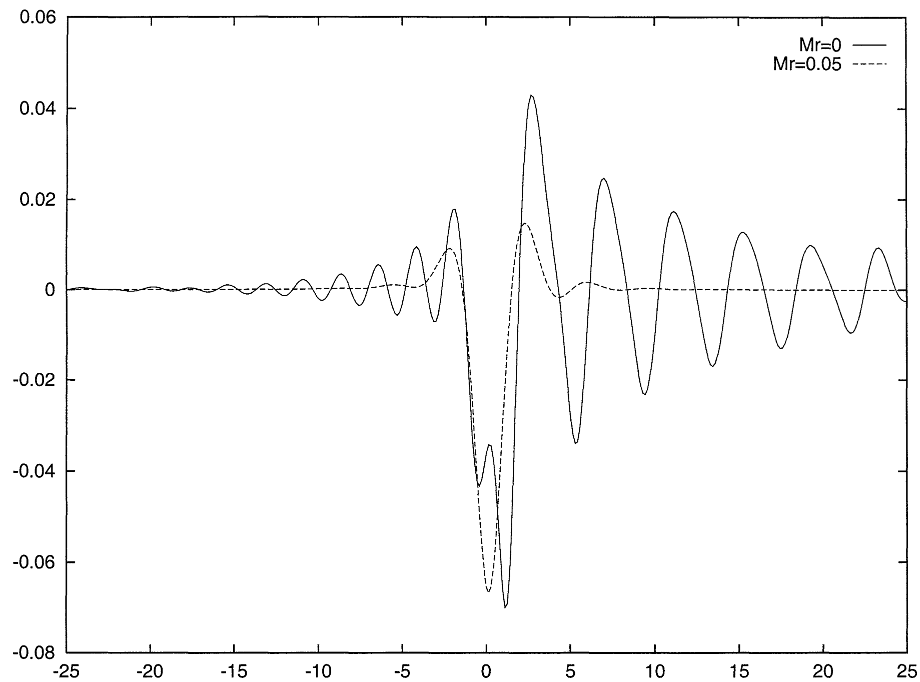

Fig. 6. Free surface profiles with $Mr = 0$ and $Mr = 0.05$ for $Re = 267$, $\alpha = 0.23$, $a = 0.02$, $b = 0.3$ and $Bo = 0$.

As mentioned earlier, the surface vorticity ω_s can be calculated from (116). Typical values of of ω_s versus x are shown in Fig. 8. These results reveal a complex oscillatory behavior for $|x|$ small.

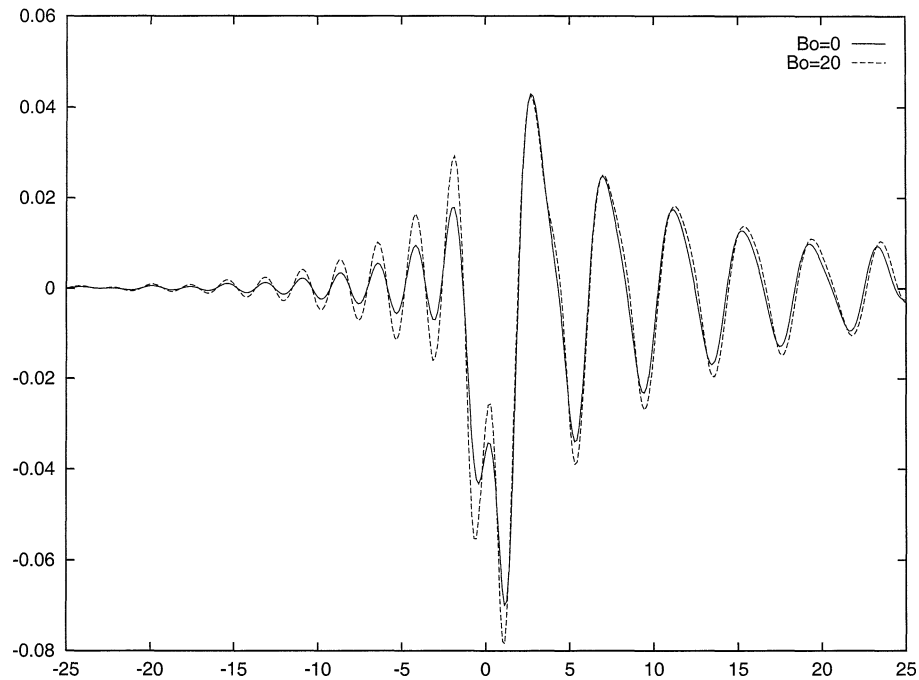

Fig. 7. Free surface profiles with $Bo = 0$ and $Bo = 20$ for $Re = 267$, $\alpha = 0.23$, $a = 0.02$, $b = 0.3$ and $Mr = 0$.

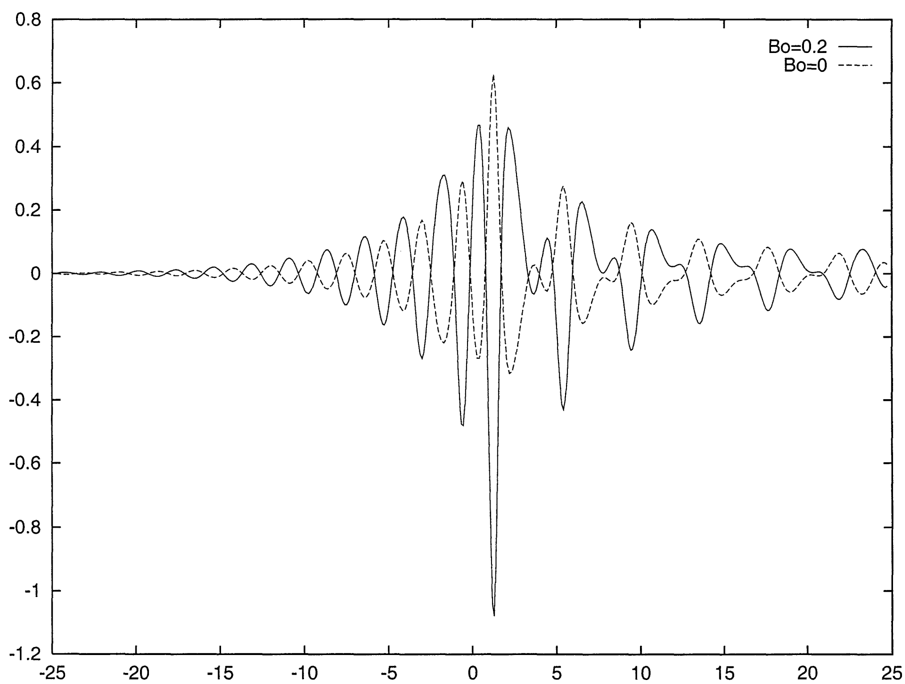

Fig. 8. Values of the surface vorticity ω_s versus x with $Bo = 0$ and $Bo = 0.2$ for $a = 0.02$, $b = 0.3$, $Re = 267$ and $Mr = 0$.

Finally let us mention that the numerical solutions for $Re \neq \infty$ were obtained without imposing explicitly a radiation condition. This is to be contrasted with the case $Re = \infty$ where a radiation condition (requiring that there is no supply of energy from infinity) needs to be imposed.

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